

A COMBINATORIAL DESCRIPTION OF HOMOTOPY GROUPS OF SPHERES*

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ABSTRACT. We give a combinatorial description of general homotopy groups of k -dimensional spheres with $k \geq 3$ as well as those of Moore spaces. For $n > k \geq 3$, we construct a finitely generated group defined by explicit generators and relations, whose center is exactly $\pi_n(S^k)$.

1. INTRODUCTION

The purpose of this article is to give an explicit combinatorial description of general homotopy groups of k -dimensional spheres with $k \geq 3$ as well as those of Moore spaces. The description is given by identifying the homotopy groups as the center of a quotient group of the self free products with amalgamation of pure braid groups by certain symmetric commutator subgroups.

A combinatorial description of $\pi_*(S^2)$ was discovered by the second author in 1994 and given in his thesis [20], with a published version in [22]. This description can be briefly summarized as follows. Let F_n be a free group of rank $n \geq 1$ with a basis given by $\{x_1, \dots, x_n\}$. Let $R_i = \langle x_i \rangle^{F_n}$ be the normal closure of x_i in F_n for $1 \leq i \leq n$. Let $R_{n+1} = \langle x_1 x_2 \cdots x_n \rangle^{F_n}$ be the normal closure of the product element $x_1 x_2 \cdots x_n$ in F_n . We can form a symmetric commutator subgroup

$$[R_1, R_2, \dots, R_{n+1}]_S = \prod_{\sigma \in \Sigma_{n+1}} [\dots [R_{\sigma(1)}, R_{\sigma(2)}], \dots, R_{\sigma(n+1)}].$$

This gives an explicit subgroup of F_n with a set of generators that can be understood by taking a collection of iterated commutators. By [22, Theorem 1.4], we have the following combinatorial description on $\pi_*(S^2)$.

Theorem 1.1. *For $n \geq 1$, there is an isomorphism*

$$\pi_{n+1}(S^2) \simeq \frac{R_1 \cap \cdots \cap R_{n+1}}{[R_1, \dots, R_{n+1}]_S}$$

Moreover, the homotopy group $\pi_{n+1}(S^2)$ is isomorphic to the center of the group $F_n/[R_1, R_2, \dots, R_{n+1}]_S$. □

The groups $F_n/[R_1, R_2, \dots, R_{n+1}]_S$ can be defined using explicit generators and relations. This situation is very interesting from the group-theoretical point of view: we don't know how to describe homotopy groups $\pi_*(S^2)$ in terms of generators and relations, but we can describe a bigger group whose center is exactly $\pi_*(S^2)$.

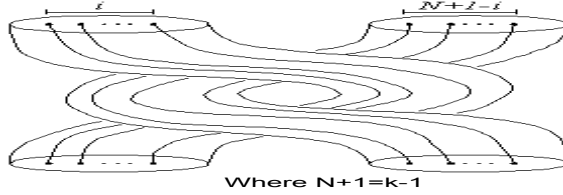
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It has been the concern of many people whether one can give a combinatorial description of homotopy groups of higher dimensional spheres, ever since the above result was announced in 1994. Technically the proof of this theorem was obtained by determining the Moore boundaries of Milnor's $F[K]$ -construction [17] on the simplicial 1-sphere S^1 , which is a simplicial group model for ΩS^2 . A canonical approach is to study Milnor's construction $F[S^k] \simeq \Omega S^{k+1}$ for $k > 1$. Although there have been some attempts [23] to study this question using $F[S^k]$, technical difficulties arise in handling Moore boundaries of $F[S^k]$ in a good way, and combinatorial descriptions of homotopy groups of higher dimensional spheres using simplicial group model $F[S^k]$ would be very messy.

In this article, we give a combinatorial description of $\pi_*(S^k)$ for any $k \geq 3$ by using free product with amalgamation of pure braid groups. Our construction is as follows. Given $k \geq 3$, $n \geq 2$, let P_n be the n -strand Artin pure braid group with the standard generators $A_{i,j}$ for $1 \leq i < j \leq n$. We construct a subgroup $Q_{n,k}$ of P_n from cabling as follows. Our cabling process starts from $P_2 = \mathbb{Z}$ generated by the 2-strand pure braid $A_{1,2}$.

Step 1. Consider the 2-strand pure braid $A_{1,2}$. Let x_i be $(k-1)$ -strand braid obtained by inserting i parallel strands into the tubular neighborhood of the first strand of $A_{1,2}$ and $k-i-1$ parallel strands into the tubular neighborhood of the second strand of $A_{1,2}$ for $1 \leq i \leq k-2$. The picture of x_i is as follows:



Step 2. Let $\alpha_k = [\dots [[x_1^{-1}, x_1 x_2^{-1}], x_2 x_3^{-1}], \dots, x_{k-3} x_{k-2}^{-1}, x_{k-2}]$ be the $(k-1)$ -strand braid.

Step 3. By applying the cabling process as in Step 1 to the element α_k , we obtain the n -strand braids y_j for $1 \leq j \leq \binom{n-1}{k-2}$.

Let $Q_{n,k}$ be the subgroup of P_n generated by y_j for $1 \leq j \leq \binom{n-1}{k-2}$. Now consider the free product with amalgamation

$$P_n *_{Q_{n,k}} P_n.$$

Let $A_{i,j}$ be the generators for the first copy of P_n and let $A'_{i,j}$ denote the generators $A_{i,j}$ for the second copy of P_n . Let $R_{i,j} = \langle A_{i,j}, A'_{i,j} \rangle^{P_n *_{Q_{n,k}} P_n}$ be the normal closure of $A_{i,j}, A'_{i,j}$ in $P_n *_{Q_{n,k}} P_n$. Let

$$[R_{i,j} \mid 1 \leq i < j \leq n]_S = \prod_{\{1,2,\dots,n\}=\{i_1,j_1,\dots,i_t,j_t\}} [[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}]$$

be the product of all commutator subgroups such that each integer $1 \leq j \leq n$ appears as one of indices at least once. By Lemma 3.6, this product can be given by taking over those commutator subgroups $[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}]$ such that

- 1) $\{i_1, j_1, \dots, i_t, j_t\} = \{1, 2, \dots, n\}$ and
- 2) $\{i_1, j_1, \dots, i_t, j_t\} \setminus \{i_p, j_p\} \neq \{1, 2, \dots, n\}$.

Our main theorem is as follows:

Theorem 1.2. *Let $k \geq 3$. The homotopy group $\pi_n(S^k)$ is isomorphic to the center of the group*

$$(P_n *_{Q_{n,k}} P_n) / [R_{i,j} \mid 1 \leq i < j \leq n]_S$$

for any n if $k > 3$ and any $n \neq 3$ if $k = 3$.

Note. The only exceptional case is that $k = 3$ and $n = 3$. In this case, $\pi_3(S^3) = \mathbb{Z}$ while the center of the group is bigger than \mathbb{Z} .

The center of the group $(P_n *_{Q_{n,k}} P_n) / [R_{i,j} \mid 1 \leq i < j \leq n]_S$ is in fact given by Brunnian-type braids in the following sense: Let $\bar{d}_k: P_n \rightarrow P_{n-1}$ be the operation of removing the k -th strand for $1 \leq k \leq n$. A Brunnian braid means an n -braid β such that $\bar{d}_k \beta = 1$ for any $1 \leq k \leq n$. Namely β becomes a trivial braid after removing any one of its strands. This notion can canonically be extended to free products of braid groups. In other words, we have a canonical operation $\bar{d}_k: P_n * P_n \rightarrow P_{n-1} * P_{n-1}$ which is a group homomorphism such that, for each n -braid β in the first copy of P_n or the second copy of P_n , $\bar{d}_k \beta$ is the $(n-1)$ -strand braid given by removing the k -th strand of β . A Brunnian-type word means a word w such that $\bar{d}_k w = 1$ for any $1 \leq k \leq n$. Without taking amalgamation, it can be seen from our techniques that the Brunnian-type braids are exactly given by the symmetric commutator subgroup $[R_{i,j} \mid 1 \leq i < j \leq n]_S$. However the question on determining Brunnian-type braids after taking amalgamation becomes very tricky. The question here is about the self free product of P_n with the amalgamation given by the subgroup $Q_{n,k}$. It is straightforward to check that the strand-removing operation \bar{d}_k maps $Q_{n,k}$ into $Q_{n-1,k}$ and so the removing operation $\bar{d}_k: P_n *_{Q_{n,k}} P_n \rightarrow P_{n-1} *_{Q_{n-1,k}} P_{n-1}$ is a well-defined group homomorphism. From our construction of simplicial groups given by free products with amalgamation, the Brunnian-type braids in $P_n *_{Q_{n,k}} P_n$ are exactly the Moore cycles in our simplicial group model for ΩS^k and so the center¹

$$Z((P_n *_{Q_{n,k}} P_n) / [R_{i,j} \mid 1 \leq i < j \leq n]_S) \cong \pi_n(S^k)$$

is exactly given by the Brunnian-type braids in $P_n *_{Q_{n,k}} P_n$ modulo the subgroup $[R_{i,j} \mid 1 \leq i < j \leq n]_S$. One important point concerning Brunnian-type braids of the self free product with amalgamation of P_n is that the homotopy groups $\pi_n(S^k)$ can be given as quotient groups for any $k \geq 3$.

¹For a group G , we denote its center by $Z(G)$.

Mark Mahowald asked in 1995 whether one can give a combinatorial description of the homotopy groups of the suspensions of real projective spaces. In this article, we also give a combinatorial description of the homotopy groups of Moore spaces as the first step for attacking Mahowald's question. Let $M(\mathbb{Z}/q, k)$ be the $(k+1)$ -dimensional Moore space. Namely $M(\mathbb{Z}/q, k) = S^k \cup_q e^{k+1}$ is the homotopy cofibre of the degree q map $S^k \rightarrow S^k$. If $k \geq 3$, we give a combinatorial description of $\pi_*(M(\mathbb{Z}/q, k))$ given as the centers of quotient groups of threefold self free product with amalgamation of pure braid groups, which is similar to the description given in Theorem 1.2. (The detailed description will be given in Section 4.) This description is less explicit than the one given in Theorem 1.2, but it leads to combinatorial descriptions of homotopy groups of finite complexes from iterated self free products with amalgamations of pure braid groups.

For the homotopy groups of 3-dimensional Moore spaces, there is an explicit combinatorial description that deserves to be described here as it arises in certain divisibility questions concerning braids. Let x_1, \dots, x_{n-1} be n -strand braid obtained by cabling $A_{1,2}$ as described in step 1 of the construction for the group $Q_{n,k}$. It was proved in [5] that the subgroup of P_n generated by x_1, \dots, x_{n-1} is a free group of rank $n-1$ with a basis given by x_1, \dots, x_{n-1} . Let $F_{n-1} = \langle x_1, \dots, x_{n-1} \rangle \leq P_n$ be the subgroup generated by x_1, \dots, x_{n-1} . Given an integer q , since $F_{n-1} = \langle x_1, \dots, x_{n-1} \rangle$ is free, there is a group homomorphism $\phi_q: F_{n-1} \rightarrow F_{n-1}$ such that $\phi_q(x_j) = x_j^q$ for $1 \leq j \leq n-1$. Now we form a free product with amalgamation by the push-out diagram

$$\begin{array}{ccc} F_{n-1} & \hookrightarrow & P_n \\ \downarrow \phi_q & & \downarrow \\ F_{n-1} & \longrightarrow & P_n *_{\phi_q} F_{n-1}, \end{array}$$

namely the group $P_n *_{\phi_q} F_{n-1}$, which is the free product by identifying the subgroup F_{n-1} with the subgroup of F_{n-1} generated by x_1^q, \dots, x_{n-1}^q in a canonical way. Let y_j denote the generator x_j for F_{n-1} as the second factor in the free product $P_n *_{\phi_q} F_{n-1}$ for $1 \leq j \leq n-1$. Let

$$R_1 = \langle y_1 \rangle^{P_n *_{\phi_q} F_{n-1}}, R_j = \langle y_{j-1} y_j^{-1} \rangle^{P_n *_{\phi_q} F_{n-1}}, R_n = \langle y_{n-1} \rangle^{P_n *_{\phi_q} F_{n-1}}$$

be the normal closure of $y_1, y_{j-1} y_j^{-1}, y_{n-1}$ in $P_n *_{\phi_q} F_{n-1}$, respectively, for $2 \leq j \leq n-1$. Let $R_{s,t} = \langle A_{s,t} \rangle^{P_n *_{\phi_q} F_{n-1}}$ be the normal closure of $A_{s,t}$ in $P_n *_{\phi_q} F_{n-1}$ for $1 \leq s < t \leq n$. Define the index set $\text{Index}(R_j) = \{j\}$ for $1 \leq j \leq n$ and $\text{Index}(R_{s,t}) = \{s, t\}$ for $1 \leq s < t \leq n$. Now define the symmetric commutator subgroup

$$[R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S = \prod_{\{1,2,\dots,n\} = \bigcup_{j=1}^t \text{Index}(C_j)} [[C_1, C_2], \dots, C_t],$$

where each $C_j = R_i$ or $R_{s,t}$ for some i or (s, t) .

Theorem 1.3. *The homotopy group $\pi_n(M(\mathbb{Z}/q, 2))$ is isomorphic to the center of the group*

$$(P_n *_{\phi_q} F_{n-1}) / [R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S$$

for $n \neq 3$.

Note. For the exceptional case $n = 3$, $\pi_3(M(\mathbb{Z}/q, 2))$ is contained in the center but the equality fails.

Some remarks concerning the methodology of this paper are given next. The notion of simplicial sets and simplicial groups have been largely studied since it was introduced in the early of 1950s, when D. Kan established the foundational work for simplicial homotopy theory [11, 12]. Various important results have been achieved by studying simplicial groups. For instance, the Adams spectral sequence can be obtained from the lower central series of Kan's construction [3] for computational purpose on homotopy groups. A combinatorial description of general homotopy groups of S^2 was given in [22] with important progress in connecting to Brunnian braids [2]. This description was generalized in [8] by studying van Kampen-type theorem for higher homotopy groups. Serious study of Brunnian braids [1, 15] introduced the notion of symmetric commutator subgroups in determining the group of Brunnian braids on surfaces S for $S \neq S^2$ or \mathbb{RP}^2 . By using this notion together with the embedding theorem in [5, Theorem 1.2] as well as the Whitehead Theorem on free products with amalgamation of simplicial groups [13, Proposition 4.3], we are able to control the Moore boundaries of our simplicial group models for the loop spaces of spheres and Moore spaces, which leads to our results.

Theorems 1.2 and 1.3 have more theoretical significance rather than computational purpose. It addresses the importance and complexity on the questions concerning Brunnian-type braids in free products with amalgamation of braid groups.

The article is organized as follows. In Section 2, we study free products with amalgamation of simplicial groups. In some cases, these products present simplicial models for loop spaces of homotopy push-out spaces. In Section 3, for $k \geq 3$, we construct simplicial groups $\mathcal{T}(S^k; \alpha)$ such that there is a homotopy equivalence $|\mathcal{T}(S^k; \alpha)| \simeq \Omega S^k$. There is a natural way to describe Moore boundaries of $\mathcal{T}(S^k; \alpha)$ and this description is a key point in the proof of Theorem 1.2 which we give in Section 3. In Section 4, we consider triple free products with amalgamation of simplicial braid groups and construct simplicial models for loop spaces for Moore spaces. For $k \geq 3$, we give a description of a finitely-generated group such that its center is $\pi_n(M(\mathbb{Z}/q, k))$ (Theorem 4.4). Section 5 is about 3-dimensional Moore spaces. In this case, the simplicial models for loop spaces of Moore spaces can be simplified. We prove Theorem 1.3 in Section 5.

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2. FREE PRODUCTS WITH AMALGAMATION ON SIMPLICIAL GROUPS

Let $\phi: G \rightarrow G'$ and $\psi: G \rightarrow G''$ be group monomorphisms. Then we have the free product with amalgamation $G' *_G G''$. More precisely $G' *_G G''$ is the quotient group of the free product $G' * G''$ by the normal closure of the elements $\phi(g)\psi(g)^{-1}$ for $g \in G$. The group $G' *_G G''$ has the universal property that the following

diagram

$$\begin{array}{ccc}
 G & \xrightarrow{\phi} & G' \\
 \downarrow \psi & & \downarrow \\
 G'' & \longrightarrow & G' *_G G''
 \end{array}$$

is a pushout diagram in the category of groups. Let $G' = \langle X' \mid R' \rangle$ and $G'' = \langle X'' \mid R'' \rangle$ be presentations of the groups G' and G'' , respectively. Let X be a set of generators for the group G . Then the group $G' *_G G''$ has a presentation

$$G' *_G G'' = \langle X', X'' \mid R', R'', \phi(x)\psi(x)^{-1} \text{ for } x \in X \rangle.$$

In particular, if X', X'', R', R'' and X are finite sets, then $G' *_G G''$ is a finitely presented group with a presentation given as above. The notion of free product with amalgamation can be canonically extended to the category of simplicial groups.

Recall that a simplicial group G consists in a sequence of groups $G = \{G_n\}_{n \geq 0}$ with face homomorphisms $d_i: G_n \rightarrow G_{n-1}$ and degeneracy homomorphisms $s_i: G_n \rightarrow G_{n+1}$ for $0 \leq i \leq n$ such that the following simplicial identities holds:

- 1) Δ -identity: $d_i d_j = d_j d_{i+1}$ for $i \geq j$,
- 2) Degeneracy Identity: $s_i s_j = s_{j+1} s_i$ for $i \leq j$,
- 3) Mixing Relation:

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ \text{id} & \text{if } i = j, j+1, \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

A simplicial homomorphism $f: G \rightarrow G'$ consists in a sequence of group homomorphism $f = \{f_n\}$ with $f_n: G_n \rightarrow G'_n$ such that $d_i^{G'} f_n = f_{n-1} d_i^G$ and $s_i^{G'} f_n = f_{n+1} s_i^G$ for $0 \leq i \leq n$. A simplicial monomorphism $f: G \rightarrow G'$ means a simplicial homomorphism $f = \{f_n\}$ such that each $f_n: G_n \rightarrow G'_n$ is a monomorphism. Similarly we have the notion of simplicial epimorphism.

For a simplicial group G , recall that the Moore chain complex $N_* G$ is defined by

$$N_n G = \bigcap_{j=1}^n \text{Ker}(d_i: G_n \rightarrow G_{n-1})$$

with the differential given by the restriction of the first face $d_0|: N_n G \rightarrow N_{n-1} G$. The Moore chain complex functor has the following important properties. For a simplicial set X , let $|X|$ denote its geometric realization.

Proposition 2.1. *The following statements hold:*

- 1) *Let G be any simplicial group. Then there is a natural isomorphism*

$$H_n(N_* G; d_0|) \cong \pi_n(|G|)$$

for all n .

- 2) *Let $f: G \rightarrow G'$ be a simplicial homomorphism. Then f is a simplicial monomorphism (epimorphism) if and only if*

$$N(f): N_q G \longrightarrow N_q G'$$

is a monomorphism (epimorphism) for all q .

3) A sequence of simplicial groups

$$1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$$

is short exact if and only if the corresponding sequence of Moore chain complexes

$$1 \rightarrow N_*G' \rightarrow N_*G \rightarrow N_*G'' \rightarrow 1$$

is short exact.

Proof. Assertion (1) is the classical theorem of John Moore, see the survey paper [7]. Assertion (2) is given in Quillen's book [18, Lemma 5, 3.8].

(3). By [2, Proposition 4.1.4], the Moore chain functor is an exact functor. We show that the inverse statement is also true. Namely if $1 \rightarrow N_*G' \rightarrow N_*G \rightarrow N_*G'' \rightarrow 1$ is short exact, then $1 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 1$ is short exact. By assertion (2), $G' \rightarrow G$ is a simplicial monomorphism and $G \rightarrow G''$ is a simplicial epimorphism. From Conduché's decomposition theorem of simplicial groups [6], the composite $G' \rightarrow G \rightarrow G''$ is trivial and so G' is mapped into $\text{Ker}(G \rightarrow G'')$. Since $N_*(G') \cong N_*\text{Ker}(G \rightarrow G'') = \text{Ker}(N_*G \rightarrow N_*G'')$, $G' \rightarrow \text{Ker}(G \rightarrow G'')$ is an isomorphism by assertion (2) and the result follows. \square

Let $Z_n G = \bigcap_{j=0}^n \text{Ker}(d_j: G_n \rightarrow G_{n-1}) \leq N_n G$ be the *Moore cycles* and let $B_n G = d_0(N_{n+1} G) \leq Z_n G$ be the *Moore boundaries*. By assertion (1), the homotopy group $\pi_n(|G|)$ is given by $Z_n G / B_n G$.

The construction of free product with amalgamation on simplicial groups is given in the same way. Let $\phi: G \rightarrow G'$ and $\psi: G \rightarrow G''$ be simplicial monomorphisms. Then $G' *_G G''$ is a simplicial group with each $(G' *_G G'')_n$ is the free product with amalgamation of $G'_n *_{G_n} G''_n$ for the group homomorphisms $\phi_n: G_n \rightarrow G'_n$ and $\psi_n: G_n \rightarrow G''_n$. The face homomorphisms are (uniquely) determined by the pushout property:

$$\begin{array}{ccccc}
 G_n & \xhookrightarrow{\phi_n} & G'_n & & \\
 \downarrow d_i^G & \searrow \partial_i^G & \downarrow d_i^{G'} & \searrow & \\
 & & G''_n & \xrightarrow{\quad} & G'_n *_{G_n} G''_n \\
 & & \downarrow d_i^{G''} & & \downarrow d_i^{G' *_G G''} \\
 G_{n-1} & \xhookrightarrow{\phi_{n-1}} & G'_{n-1} & & \\
 \downarrow d_i^G & \searrow \partial_i^G & \downarrow d_i^{G''} & \searrow & \\
 & & G''_{n-1} & \xrightarrow{\quad} & G'_{n-1} *_{G_{n-1}} G''_{n-1}
 \end{array}$$

Similarly the degeneracy homomorphisms are (uniquely) determined by the pushout property. The uniqueness of the induced face and degeneracy homomorphisms forces the simplicial identities to hold for $d_i^{G' *_G G''}$ and $s_j^{G' *_G G''}$ and so $G' *_G G''$ becomes a simplicial group. If we write the elements w in $(G' *_G G'')_n = G'_n *_{G_n} G''_n$ in terms of words as a product of elements from G'_n or G''_n , then $d_i^{G' *_G G''}(w)$ is given

by applying $d_i^{G'}$ or $d_i^{G''}$ to the factors of w . Similarly we can compute degeneracy homomorphism $s_i^{G' *_{G'} G''}$ on $(G' *_G G'')_n$ in the same manner.

There is a classifying space functor from the category of simplicial groups to the category of simplicial sets, denoted by \bar{W} , with the property that the geometric realization of $\bar{W}(G)$ is a classifying space of the geometric realization of the simplicial group G . We refer to Curtis' paper [7] for the detailed construction of the functor \bar{W} .

An important property of free product with amalgamation on simplicial groups is that the classifying space of $G' *_G G''$ can be controlled. This property is a simplicial consequence of the classical asphericity result of J. H. C. Whitehead [19, Theorem 5] in 1939 and the formal statement of the following theorem was given in Kan-Thurston's paper [13, Proposition 4.3].

Theorem 2.2 (Whitehead Theorem). *Let $\phi: G \rightarrow G'$ and $\psi: G \rightarrow G''$ be simplicial monomorphisms. Then the classifying space $\bar{W}(G' *_G G'')$ is the homotopy push-out of the diagram*

$$\begin{array}{ccc} \bar{W}G & \xrightarrow{\bar{W}\phi} & \bar{W}G' \\ \bar{W}\psi \downarrow & \text{push} & \downarrow \\ \bar{W}G'' & \longrightarrow & \bar{W}(G' *_G G''). \end{array}$$

□

3. DESCRIPTION OF HOMOTOPY GROUPS OF SPHERES AND PROOF OF THEOREM 1.2

In this section, we are going to construct a simplicial group model $\mathcal{T}(S^k)$ for ΩS^k , $k \geq 3$, by using pure braid groups. From this, we are able to give a combinatorial description of the homotopy group $\pi_q(S^k)$ for general q .

3.1. Milnor's $F[K]$ -construction on spheres. Let K be a simplicial set with a fixed choice of base-point $s_0^n x_0 \in K_n$. Milnor [17] constructed a simplicial group $F[K]$ where $F[K_n]$ is the free group generated by K_n subject to the single relation that $s_0^n x_0 = 1$. The face and degeneracy homomorphisms on $F[K]$ are induced by the face and degeneracy functions on K . An important property of Milnor's construction is that the geometric realization $|K|$ of $F[K]$ is homotopy equivalent to $\Omega\Sigma|K|$. (Note. In Milnor's paper [17], K is required to be a reduced simplicial set. This result actually holds for any pointed simplicial set by a more general result [21, Theorem 4.9].)

We are interested in specific simplicial group models for ΩS^{k+1} and so we start by considering the simplicial k -sphere S^k . Recall that the simplicial k -simplex $\Delta[k]$ can be defined explicitly as follows:

$$\Delta[k]_n = \{(i_0, i_1, \dots, i_n) \mid 0 \leq i_0 \leq i_1 \leq \dots \leq i_n \leq k\} \text{ with } d_i: \Delta[k]_n \rightarrow \Delta[k]_{n-1} \text{ given by removing the } (i+1)\text{st coordinate and } s_i: \Delta[k]_n \rightarrow \Delta[k]_{n+1} \text{ given by doubling the } (i+1)\text{st coordinate for } 0 \leq i \leq n.$$

Let $\sigma_k = (0, 1, \dots, k) \in \Delta[k]_k$ and let $\partial\Delta[k]$ be the simplicial subset of $\Delta[k]$ generated by the faces $d_0\sigma_k, \dots, d_k\sigma_k$. Namely $\partial\Delta[k]$ is the smallest simplicial subset of $\Delta[k]$ containing $d_i\sigma_k$ for $0 \leq i \leq k$. Let $S^k = \Delta[k]/\partial\Delta[k]$. Then the geometric realization $|S^k|$ is homeomorphic to the standard k -sphere S^k . As a simplicial set, $S_n^k = \{*\}$ for $n < k$ and

$$(3.1) \quad \begin{aligned} S_n^k &= \{*, (i_0, i_1, \dots, i_n) \mid 0 \leq i_0 \leq i_1 \leq \dots \leq i_n \leq k \\ &\quad \text{with } \{0, 1, \dots, k\} = \{i_0, i_1, \dots, i_n\}\} \\ &= \{*, s_{j_{n-k}} s_{j_{n-k-1}} \dots s_{j_1} \sigma_k \mid 0 \leq j_1 < j_2 < \dots < j_{n-k} \leq n-1\} \end{aligned}$$

for $n \geq k$. In the first description above, it is required that each $0 \leq j \leq k$ appears at least once in the sequence (i_0, \dots, i_n) . In this description, we can describe the faces and degeneracies by removing-doubling coordinates where we identify the sequence (i_0, \dots, i_n) to be the base-point of any one of $0 \leq j \leq k$ does not appear in (i_0, \dots, i_n) . In the second description, we can use the simplicial identities to describe the faces and degeneracies on S^k .

By applying Milnor's construction to S^k , we obtain the simplicial group $F[S^k] \simeq \Omega S^{k+1}$ with $F[S^k]_n$ a free group of rank $\binom{n}{k}$. The generators for $F[S^k]_n$ are given in formula (3.1) with $*$ = 1.

3.2. The Simplicial Group AP_* . There is a canonical simplicial group arising from pure braid groups systematically investigated in [2]. We are only interested in classical Artin pure braids and so we follow the discussion in [5]. Let $\text{AP}_n = P_{n+1}$ with the face homomorphism

$$d_i: \text{AP}_n = P_{n+1} \longrightarrow \text{AP}_{n-1} = P_n$$

given by removing the $(i+1)$ st strand of $(n+1)$ -strand pure braids and the degeneracy homomorphism

$$s_i: \text{AP}_n = P_{n+1} \longrightarrow \text{AP}_{n+1} = P_{n+2}$$

given by doubling the $(i+1)$ st strand of $(n+1)$ -strand pure braids for $0 \leq i \leq n$. Then AP_* forms a simplicial group. Let $A_{i,j}$, $1 \leq i < j \leq n+1$, be the standard generators for $\text{AP}_n = P_{n+1}$. Then the face operations in the simplicial group AP_* are defined as follows:

$$(3.2) \quad d_t(A_{i,j}) = \begin{cases} A_{i-1,j-1} & \text{if } t+1 < i, \\ 1 & \text{if } t+1 = i, \\ A_{i,j-1} & \text{if } i < t+1 < j, \\ 1 & \text{if } t+1 = j, \\ A_{i,j} & \text{if } t+1 > j. \end{cases}$$

and the degeneracy operations are defined as follows:

$$(3.3) \quad s_t(A_{i,j}) = \begin{cases} A_{i+1,j+1} & \text{if } t+1 < i, \\ A_{i,j+1} \cdot A_{i+1,j+1} & \text{if } t+1 = i, \\ A_{i,j+1} & \text{if } i < t+1 < j, \\ A_{i,j} \cdot A_{i,j+1} & \text{if } t+1 = j, \\ A_{i,j} & \text{if } t+1 > j. \end{cases}$$

Observe that $AP_1 = P_2 \cong \mathbb{Z}$ is generated by $A_{1,2}$ with $d_0 A_{1,2} = d_1 A_{1,2} = 1$. The representing simplicial map

$$f_{A_{1,2}}: S^1 \longrightarrow AP_*$$

with $f_{\sigma_1} = A_{1,2}$ extends uniquely to a simplicial homomorphism

$$\Theta: F[S^1] \longrightarrow AP_*.$$

The following embedding theorem plays an important role for our constructions of simplicial group models for the loop spaces of spheres and Moore spaces.

Theorem 3.1. [5, Theorem 1.2] *The simplicial homomorphism*

$$\Theta: F[S^1] \longrightarrow AP_*$$

is a simplicial monomorphism. □

3.3. Simplicial Group Models for ΩS^k with $k \geq 3$. Assume that $k \geq 3$. Let $\alpha \in F[S^1]_{k-2}$ such that

- 1) $\alpha \neq 1$ and
- 2) $d_j \alpha = 1$ for all $0 \leq j \leq k-2$, that is, α is a Moore cycle.

(**Note.** We do not assume that α induces a nontrivial element in $\pi_{k-2}(F[S^1]) = \pi_{k-1}(S^2)$. There are many choices for such an α . We will give a particular choice of α with braided instructions later. For a moment α is given by any nontrivial Moore cycle.) The representing simplicial map

$$f_\alpha: S^{k-2} \longrightarrow F[S^1]$$

extends uniquely to a simplicial homomorphism

$$\tilde{f}_\alpha: F[S^{k-2}] \longrightarrow F[S^1]$$

by the universal property of Milnor's construction.

Lemma 3.2. *Let $k \geq 3$ and let $\alpha \neq 1 \in F[S^1]_{k-2}$ be a Moore cycle. Then the map*

$$\tilde{f}_\alpha: F[S^{k-2}] \longrightarrow F[S^1]$$

is a simplicial monomorphism.

Proof. Let $G = \tilde{f}_\alpha(F[S^{k-2}])$ be the image of \tilde{f}_α . Then G is a simplicial subgroup of $F[S^1]$. Since $F[S^1]_q$ is a free group, G_q is free group for each q . The statement will follow if we can prove that the simplicial epimorphism

$$\tilde{f}_\alpha: F[S^{k-2}] \longrightarrow G$$

is a simplicial monomorphism. Observe that since each $F[S^{k-2}]_q$ is a free group which is residually nilpotent, it suffices to show that the morphism of the associated Lie algebras induced from the lower central series

$$L(\tilde{f}_\alpha): L(F[S^{k-1}]) \longrightarrow L(G)$$

is a simplicial isomorphism. For each q , since both $F[S^{k-2}]_q$ and G_q are free group, their associated Lie algebras are the free Lie algebras generated by their abelianizations. Thus it suffices to show that

$$\tilde{f}_\alpha^{\text{ab}}: F[S^{k-2}]^{\text{ab}} = K(\mathbb{Z}, k-2) \longrightarrow G^{\text{ab}}$$

is a simplicial isomorphism.

Note that the Moore chain complex of $K(\mathbb{Z}, k-2)$ is given by

$$N_q K(\mathbb{Z}, k-2) = \begin{cases} 0 & \text{if } q \neq k-2, \\ \mathbb{Z} & \text{if } q = k-2. \end{cases}$$

Since $\tilde{f}_\alpha^{\text{ab}}: K(\mathbb{Z}, k-2) \rightarrow G^{\text{ab}}$ is a simplicial epimorphism,

$$N(\tilde{f}_\alpha^{\text{ab}}): N_q K(\mathbb{Z}, k-2) \longrightarrow N_q G^{\text{ab}}$$

is an epimorphism for any q by Proposition 2.1. It follows that $N_q G^{\text{ab}} = 0$ for $q \neq k-2$. For $q = k-2$, we have $N_{k-2} G^{\text{ab}} = G_{k-2} = \langle \alpha \rangle \cong \mathbb{Z}$ with

$$N(\tilde{f}_\alpha^{\text{ab}}): N_{k-2} K(\mathbb{Z}, k-2) \cong \mathbb{Z} \longrightarrow N_{k-2} G^{\text{ab}} \cong \mathbb{Z}$$

an isomorphism from the definition of f_α . Thus

$$N(\tilde{f}_\alpha^{\text{ab}}): NF[S^{k-2}]^{\text{ab}} \longrightarrow NG^{\text{ab}}$$

is an isomorphism. By Proposition 2.1, $\tilde{f}_\alpha^{\text{ab}}: F[S^{k-2}]^{\text{ab}} = K(\mathbb{Z}, k-2) \longrightarrow G^{\text{ab}}$ is a simplicial isomorphism. This finishes the proof. \square

Now, by Theorem 3.1 and Lemma 3.2, the composite

$$\phi_\alpha: F[S^{k-2}] \xrightarrow{\tilde{f}_\alpha} F[S^1] \xrightarrow{\Theta} \text{AP}_*$$

is a simplicial monomorphism. Define the simplicial group $\mathcal{T}(S^k; \alpha)$ to be the free product with amalgamation defined by the diagram

$$\begin{array}{ccc} F[S^{k-2}] & \xrightarrow{\phi_\alpha} & \text{AP}_* \\ \downarrow \phi_\alpha & & \downarrow \\ \text{AP}_* & \longrightarrow & \mathcal{T}(S^k; \alpha) = \text{AP}_* *_F[S^{k-2}] \text{AP}_*. \end{array}$$

Theorem 3.3. *Let $k \geq 3$ and let $\alpha \neq 1 \in F[S^1]_{k-2}$ be a Moore cycle. Then the geometric realization of the simplicial group $\mathcal{T}(S^k; \alpha)$ is homotopy equivalent to ΩS^k .*

Proof. By Theorems 2.2, the classifying space $\bar{W}\mathcal{T}(S^k; \alpha)$ is the homotopy push-out of

$$\begin{array}{ccc} \bar{W}F[S^{k-2}] \simeq S^{k-1} & \longrightarrow & \bar{W}\text{AP}_* \\ \downarrow & & \downarrow \\ \bar{W}\text{AP}_* & \longrightarrow & \bar{W}\mathcal{T}(S^k; \alpha). \end{array}$$

By [5, Theorem 1.1], AP_* is a contractible simplicial group and so $\bar{W}\text{AP}_*$ is contractible. It follows that

$$\bar{W}\mathcal{T}(S^k; \alpha) \simeq S^k$$

and hence the result. \square

3.4. Some Technical Lemmas. Recall [16, p. 288-289] that a *bracket arrangement of weight n* in a group G is a map $\beta^n: G^n \rightarrow G$ which is defined inductively as follows:

$$\beta^1 = \text{id}_G, \quad \beta^2(a_1, a_2) = [a_1, a_2]$$

for any $a_1, a_2 \in G$, where $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$. Suppose that the bracket arrangements of weight k are defined for $1 \leq k < n$ with $n \geq 3$. A map $\beta^n: G^n \rightarrow G$ is called a bracket arrangement of weight n if β^n is the composite

$$G^n = G^k \times G^{n-k} \xrightarrow{\beta^k \times \beta^{n-k}} G \times G \xrightarrow{\beta^2} G$$

for some bracket arrangements β^k and β^{n-k} of weight k and $n-k$, respectively, with $1 \leq k < n$. For instance, if $n = 3$, there are two bracket arrangements given by $[[a_1, a_2], a_3]$ and $[a_1, [a_2, a_3]]$.

Let R_j be a sequence of subgroups of G for $1 \leq j \leq n$. The *fat commutator subgroup* $[[R_1, R_2, \dots, R_n]]$ is defined to be the subgroup of G generated by all of the commutators

$$\beta^t(g_{i_1}, \dots, g_{i_t}),$$

where

- 1) $1 \leq i_s \leq n$;
- 2) $\{i_1, \dots, i_t\} = \{1, \dots, n\}$, that is each integer in $\{1, 2, \dots, n\}$ appears as at least one of the integers i_s ;
- 3) $g_j \in R_j$;
- 4) β^t runs over all of the bracket arrangements of weight t (with $t \geq n$).

For convenience, let $[[R_1]] = R_1$.

The *symmetric commutator subgroup* $[R_1, R_2, \dots, R_n]_S$ defined by

$$[R_1, R_2, \dots, R_n]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}, \dots, R_{\sigma(n)}],$$

where $[[R_{\sigma(1)}, R_{\sigma(2)}, \dots, R_{\sigma(n)}]]$ is the subgroup generated by the left iterated commutators

$$[[[g_1, g_2], g_3], \dots, g_n]$$

with $g_i \in R_{\sigma(i)}$. For convenience, let $[R_1]_S = R_1$. From the definition, the symmetric commutator subgroup is a subgroup of the fat commutator subgroup. In fact they are the same subgroup by the following theorem provided that each R_j is normal.

Lemma 3.4. [15, Theorem 1.1] *Let R_j be any normal subgroup of a group G with $1 \leq j \leq n$. Then*

$$[[R_1, R_2, \dots, R_n]] = [R_1, R_2, \dots, R_n]_S.$$

□

One can determine the Moore chains and boundaries for the self free products of AP_* with a help of the Kurosh theorem on the structure of subgroups of free products. However, in order to get this description, we will use another method. We construct a simplicial free group \mathcal{G} as follows: For each $n \geq 0$, the group \mathcal{G}_n is the free group generated by $x_{i,j}$ for $1 \leq i < j \leq n+1$. The face and degeneracy operations are given by formulae (3.2) and (3.3), where we replace $A_{i,j}$ by $x_{i,j}$. It is straightforward to check that the simplicial identities hold. Thus we have a simplicial group \mathcal{G} .

Now we are going to determine the Moore chains and Moore cycles of the free products of \mathcal{G} . Let J be an index set and let $\mathcal{G}^{*J} = *_{\alpha \in J} \mathcal{G}(\alpha)$, where each $\mathcal{G}(\alpha)$ is a copy of \mathcal{G} indexed by an element $\alpha \in J$. For each group $(\mathcal{G}(\alpha)_n = \mathcal{G}_n)$, let $x_{i,j}(\alpha)$ denote the generator $x_{i,j}$ for $1 \leq i < j \leq n+1$. From the definition, $\mathcal{G}_n^{*J} = *_{\alpha \in J} \mathcal{G}(\alpha)_n$ is a free group with a basis given by $\{x_{i,j}(\alpha) \mid 1 \leq i < j \leq n+1, \alpha \in J\}$.

A *basic word* in the group \mathcal{G}_n^{*J} means one of the elements $x_{i,j}(\alpha)^{\pm 1}$ for some $\alpha \in J$ and so $1 \leq i < j \leq n+1$. Let

$$w = \beta_t(x_{i_1,j_1}(\alpha_1)^{\pm 1}, x_{i_2,j_2}(\alpha_2)^{\pm 1}, \dots, x_{i_t,j_t}(\alpha_t)^{\pm 1})$$

be a t -fold iterated commutator on basic words, where the bracket $\beta_t(\dots)$ is any bracket arrangement. Define

$$\text{Index}(w) = \{i_1, j_1, i_2, j_2, \dots, i_t, j_t\} \subseteq \{1, 2, \dots, n+1\}.$$

(**Note.** In our definition, $\text{Index}(w)$ is only well-defined for commutators with entries from basic words.)

For each pair $1 \leq i < j \leq n+1$, let

$$R_{i,j}^J = \langle x_{i,j}(\alpha) \mid \alpha \in J \rangle^{\mathcal{G}^{*J}}$$

be the normal closure of the elements $x_{i,j}(\alpha)$, $\alpha \in J$, in the group \mathcal{G}^{*J} . For a subset $T \subseteq \{1, 2, \dots, n+1\}$, define

$$R[T] = \prod_{T \subseteq \{i_1, j_1, i_2, j_2, \dots, i_t, j_t\}} [[R_{i_1, j_1}, R_{i_2, j_2}], \dots, R_{i_t, j_t}]$$

be the product of the iterated commutator subgroup of $R_{i,j}$'s such that each number in T occurs at least once in the indices of $R_{i,j}$'s. (Here if $t = 1$, then we let commutator subgroup $[R_{i_1, j_1}] = R_{i_1, j_1}$ by convention.) In the case that $T = \{1, 2, \dots, n+1\}$, we denote

$$[R_{i,j} \mid 1 \leq i < j \leq n+1]_S$$

by $R[1, 2, \dots, n+1]$.

Lemma 3.5. *Let \mathcal{G}^{*J} be the self free product of \mathcal{G} over a set J . Then*

- 1) *The Moore chains $N_n \mathcal{G}^{*J} = R[2, 3, \dots, n+1]$.*
- 2) *The Moore cycles $\mathcal{Z}_n \mathcal{G}^{*J} = R[1, 2, 3, \dots, n+1] = [R_{i,j} \mid 1 \leq i < j \leq n+1]_S$.*
- 3) *The Moore boundaries $\mathcal{B}_n \mathcal{G}^{*J} = \mathcal{Z}_n \mathcal{G}^{*J} = R[1, 2, 3, \dots, n+1] = [R_{i,j} \mid 1 \leq i < j \leq n+1]_S$.*

Proof. For assertions (1) and (2), the direction

$$R[2, 3, \dots, n+1] \leq N_n \mathcal{G}^{*J} \text{ and } R[1, 2, 3, \dots, n+1] \leq \mathcal{Z}_n \mathcal{G}^{*J}$$

can be easily checked as follows. From equation (3.2), we have $d_k x_{i,j}(\alpha) = 1$ for $\alpha \in J$ if $k+1 = i$ or j . Thus

$$R_{i,j} \leq \text{Ker}(d_k: \mathcal{G}_n^{*J} \rightarrow \mathcal{G}_{n-1}^{*J})$$

if $k+1 = i$ or j . Thus

$$[[R_{i_1, j_1}, R_{i_2, j_2}], \dots, R_{i_t, j_t}] \leq N_n \mathcal{G}^{*J} = \bigcap_{k=1}^n \text{Ker}(d_k: \mathcal{G}_n^{*J} \rightarrow \mathcal{G}_{n-1}^{*J})$$

if $\{2, 3, \dots, n+1\} \subseteq \{i_1, j_1, i_2, j_2, \dots, i_t, j_t\}$ since each d_k , $1 \leq k \leq n$, sends one of entries R_{i_s, j_s} in this (iterated) commutator subgroup to the trivial group. It follows

that $R[2, 3, \dots, n+1] \leq N_n \mathcal{G}^{*J}$. Similarly $R[1, 2, \dots, n+1] \leq \mathcal{Z}_{n+1} \mathcal{G}^{*J}$. Thus the main point is to prove that

$$(3.4) \quad N_n \mathcal{G}^{*J} \leq R[2, 3, \dots, n+1] \text{ and } \mathcal{Z}_n \mathcal{G}^{*J} \leq R[1, 2, \dots, n+1].$$

If $n = 1$, then $R[2] = R[1, 2] = \mathcal{G}_1^{*J}$ because \mathcal{G}_1^{*J} is generated by $x_{1,2}(\alpha)$ for $\alpha \in J$. In this case, the identity that $N_1 \mathcal{G}^{*J} = \mathcal{Z}_1 \mathcal{G}^{*J} = R[2] = R[1, 2] = \mathcal{G}_1^{*J}$. Thus we may assume that $n \geq 2$.

We first consider the last face operation

$$d_n: \mathcal{G}_n^{*J} \longrightarrow \mathcal{G}_{n-1}^{*J}.$$

Let $K_n = \text{Ker}(d_n: \mathcal{G}_n^{*J} \rightarrow \mathcal{G}_{n-1}^{*J})$. From equation (3.2),

$$d_n(x_{i,j}(\alpha)) = \begin{cases} 1 & \text{if } 1 \leq i < j = n+1, \\ x_{i,j}(\alpha) & \text{if } 1 \leq i < j \leq n. \end{cases}$$

Observe that the basis of \mathcal{G}_n^{*J} is given by the disjoint union of the basis of \mathcal{G}_{n-1}^{*J} with the set $\{x_{i,n+1}(\alpha) \mid 1 \leq i < n+1, \alpha \in J\}$. By [22, Proposition 3.3], a basis for the free group K_n is given by the subset X_n of \mathcal{G}_n^{*J} consisting of all of the following iterated commutators on basic words

$$(3.5) \quad w = [[[x_{i,n}(\alpha_0), x_{i_1,j_1}^{\epsilon_1}(\alpha_1)], x_{i_2,j_2}^{\epsilon_2}(\alpha_2)], \dots, x_{i_t,j_t}^{\epsilon_t}(\alpha_t)],$$

where

- 1) $t \geq 0$ (Here if $t = 0$, then $w = [x_{i,n}(\alpha_0)] = x_{i,n}(\alpha_0)$.)
- 2) $\epsilon_s = \pm 1$ for $1 \leq s \leq t$,
- 3) $1 \leq i_s < j_s \leq n$ for $1 \leq s \leq t$,
- 4) $\alpha_s \in J$ for $0 \leq s \leq t$ and
- 5) the word $x_{i_1,j_1}^{\epsilon_1}(\alpha_1)x_{i_2,j_2}^{\epsilon_2}(\alpha_2) \cdots x_{i_t,j_t}^{\epsilon_t}(\alpha_t)$ is an irreducible word in the group $\mathcal{G}_{n-1}^{*J} \leq \mathcal{G}_n^{*J}$.

Next we consider the face operation d_k restricted to K_n for $0 \leq k < n$. From the Δ -identity $d_k d_n = d_{n-1} d_k$ for $1 \leq k \leq n-1$, we have the commutative diagram of short exact sequence of groups

$$(3.6) \quad \begin{array}{ccccc} K_n = F(X_n) & \hookrightarrow & \mathcal{G}_n^{*J} & \xrightarrow{d_n} & \mathcal{G}_{n-1}^{*J} \\ \downarrow d_k|_{K_n} & & \downarrow d_k & & \downarrow d_k \\ K_{n-1} = F(X_{n-1}) & \hookrightarrow & \mathcal{G}_{n-1}^{*J} & \xrightarrow{d_{n-1}} & \mathcal{G}_{n-2}^{*J} \end{array}$$

for $1 \leq k \leq n-1$. Consider $d_k w$ for $w \in X_n$. From equation (3.2), $d_k x_{i,n}(\alpha)$ is given by the following table

$$d_k \quad \left(\begin{array}{cccccc} x_{1,n}(\alpha) & \cdots & x_{k-1,n}(\alpha) & x_{k,n}(\alpha) & x_{k+1,n}(\alpha) & \cdots & x_{n-1,n}(\alpha) \\ \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\ x_{1,n-1}(\alpha) & \cdots & x_{k-1,n-1}(\alpha) & 1 & x_{k,n-1}(\alpha) & \cdots & x_{n-2,n-1}(\alpha) \end{array} \right).$$

We now start to prove statement (3.4). Let

$$X_n(k) = \{w \in X_n \mid k+1 \in \text{Index}(w)\}$$

for $0 \leq k \leq n-1$. If $w \in X_n(k)$, then $d_k w = 1$ as d_k sends one of the entries in the commutator w to 1. Let $w \in X_n \setminus X_n(k)$ be written as in (3.5). Then $i \neq k+1$

and $k+1 \notin \{i_1, j_1, \dots, i_t, j_t\}$. From the above table, $d_k x_{i,n}(\alpha_0) = x_{i,n-1}(\alpha_0)$ for $i < k+1$ and $x_{i-1,n-1}(\alpha_0)$ for $i > k+1$. For other entries $x_{i_s, j_s}^{\epsilon_s}(\alpha_s)$, we have

$$d_k(x_{i_s, j_s}^{\epsilon_s}(\alpha_s)) = \begin{cases} x_{i_s-1, j_s-1}^{\epsilon_s}(\alpha_s) & \text{if } k+1 < i_s, \\ x_{i_s, j_s-1}^{\epsilon_s}(\alpha_s) & \text{if } i_s < k+1 < j_s, \\ x_{i_s, j_s}^{\epsilon_s}(\alpha_s) & \text{if } k+1 > j_s. \end{cases}$$

Observe that

$$d_k: \{x_{i,j}(\alpha) \mid \alpha \in J, 1 \leq i < j \leq n \text{ and } k+1 \neq i, j\} \longrightarrow \{x_{i,j}(\alpha) \mid \alpha \in J, 1 \leq i < j \leq n-1\}$$

is a bijection. The restriction of d_k in the subgroup

$$d_k|: F(x_{i,j}(\alpha) \mid \alpha \in J, 1 \leq i < j \leq n \text{ and } k+1 \neq i, j) \longrightarrow \mathcal{G}_{n-2}^{*J}$$

is an isomorphism. Since the word

$$x_{i_1, j_1}^{\epsilon_1}(\alpha_1) x_{i_2, j_2}^{\epsilon_2}(\alpha_2) \cdots x_{i_t, j_t}^{\epsilon_t}(\alpha_t) \in F(x_{i,j}(\alpha) \mid \alpha \in J, 1 \leq i < j \leq n \text{ and } k+1 \neq i, j)$$

is irreducible, the word

$$d_k(x_{i_1, j_1}^{\epsilon_1}(\alpha_1) x_{i_2, j_2}^{\epsilon_2}(\alpha_2) \cdots x_{i_t, j_t}^{\epsilon_t}(\alpha_t)) = (d_k x_{i_1, j_1}^{\epsilon_1}(\alpha_1)) (d_k x_{i_2, j_2}^{\epsilon_2}(\alpha_2)) \cdots (d_k x_{i_t, j_t}^{\epsilon_t}(\alpha_t))$$

is irreducible in $\mathcal{G}_{n-2}^{*J} \leq \mathcal{G}_{n-1}^{*J}$. It follows that $d_k w \in X_{n-1}$ for each $w \in X_n \setminus X_n(k)$ and the function

$$d_k: X_n \setminus X_n(k) \longrightarrow X_{n-1}$$

is a bijection. This allows us to apply the algorithm in [22, Section 3] to

$$d_k|: K_n = F(X_n) \longrightarrow K_{n-1} = F(X_{n-1})$$

for $0 \leq k \leq n-1$ in diagram (3.6) and so, by [22, Theorem 3.4], the Moore chains

$$N_n \mathcal{G}^{*J} = \bigcap_{k=1}^{n-1} \text{Ker}(d_k|: K_n \rightarrow K_{n-1})$$

are generated by certain iterated commutators

$$(3.7) \quad w = \beta_t(x_{i_1, j_1}(\alpha_1)^{\pm 1}, x_{i_2, j_2}(\alpha_2)^{\pm 1}, \dots, x_{i_t, j_t}(\alpha_t)^{\pm 1})$$

with $\{2, 3, \dots, n+1\} \in \text{Index}(w)$ and the Moore cycles

$$\mathcal{Z}_n \mathcal{G}^{*J} = \bigcap_{k=0}^{n-1} \text{Ker}(d_k|: K_n \rightarrow K_{n-1})$$

is generated by certain iterated commutators

$$(3.8) \quad w = \beta_t(x_{i_1, j_1}(\alpha_1)^{\pm 1}, x_{i_2, j_2}(\alpha_2)^{\pm 1}, \dots, x_{i_t, j_t}(\alpha_t)^{\pm 1})$$

with $\{1, 2, \dots, n+1\} \in \text{Index}(w)$. (**Note.** The commutator w in (3.7) or (3.8) may not be in the standard form from left to right.) Since each entry $x_{i_s, j_s}(\alpha_s)^{\pm 1}$ belongs to R_{i_s, j_s} , the commutator w in (3.7) or (3.8) lies in the fat commutator subgroup $[[R_{i_1, j_1}, R_{i_2, j_2}, \dots, R_{i_t, j_t}]]$ and so, by Lemma 3.4,

$$w \in \prod_{\sigma \in \Sigma_t} [[R_{i_{\sigma(1)}, j_{\sigma(1)}}, R_{i_{\sigma(2)}, j_{\sigma(2)}}], \dots, R_{i_{\sigma(t)}, j_{\sigma(t)}}] \leq R[s, s+1, \dots, n+1],$$

where $s = 2$ in the case of (3.7) and $s = 1$ in the case of (3.8). This finishes the proof of statement (3.4) and hence assertion (1) and (2).

(3). By assertion (2),

$$\mathcal{Z}_n \mathcal{G}^{*J} = \prod_{\{1,2,\dots,n+1\} \subseteq \{i_1,j_1,i_2,j_2,\dots,i_t,j_t\}} [[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}].$$

From equation 3.2, we have $d_0 x_{i+1,j+1}(\alpha) = x_{i,j}(\alpha)$ for $1 \leq i < j \leq n+1$ and $\alpha \in J$. Thus

$$d_0(R_{i+1,j+1}) = R_{i,j}$$

for $1 \leq i < j \leq n+1$. Given a factor $[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}]$ in $\mathcal{Z}_n \mathcal{G}^{*J}$ with $\{1, 2, \dots, n+1\} \subseteq \{i_1, j_1, i_2, j_2, \dots, i_t, j_t\}$, we have

$$d_0([R_{i_1+1,j_1+1}, R_{i_2+1,j_2+1}], \dots, R_{i_t+1,j_t+1}) = [[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}].$$

Since $\{2, 3, \dots, n+2\} \subseteq \{i_1+1, j_1+1, i_2+1, j_2+1, \dots, i_t+1, j_t+1\}$, the subgroup

$$\prod_{\{1,2,\dots,n+1\} \subseteq \{i_1,j_1,i_2,j_2,\dots,i_t,j_t\}} [[R_{i_1+1,j_1+1}, R_{i_2+1,j_2+1}], \dots, R_{i_t+1,j_t+1}] \leq N_{n+1} \mathcal{G}^{*J}$$

with

$$d_0 \left(\prod_{\{1,2,\dots,n+1\} \subseteq \{i_1,j_1,i_2,j_2,\dots,i_t,j_t\}} [[R_{i_1+1,j_1+1}, R_{i_2+1,j_2+1}], \dots, R_{i_t+1,j_t+1}] \right) = \mathcal{Z}_n \mathcal{G}^{*J}.$$

It follows that $\mathcal{Z}_n \mathcal{G}^{*J} \leq \mathcal{B}_n \mathcal{G}^{*J}$. Assertion (3) follows and this finishes the proof. \square

The following lemma states that $R[1, 2, \dots, n+1]$ can be given by the product of a finite collection of commutator subgroups.

Lemma 3.6. *The subgroup $R[1, 2, \dots, n+1]$ of \mathcal{G}_n^{*J} is the product of the following commutator subgroups*

$$[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}],$$

where

- 1) $\{i_1, j_1, i_2, j_2, \dots, i_t, j_t\} = \{1, 2, \dots, n+1\}$ and
- 2) $\{i_1, j_1, i_2, j_2, \dots, i_t, j_t\} \setminus \{i_p, j_p\} \neq \{1, 2, \dots, n+1\}$ for any $1 \leq p \leq t$.

Proof. Let H be the product of the commutator subgroups given in the statement. Clearly $H \leq R[1, 2, \dots, n+1]$. Now consider the factor

$$[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}]$$

with $\{1, 2, \dots, n+1\} = \{i_1, j_1, i_2, j_2, \dots, i_t, j_t\}$ in $R[1, 2, \dots, n+1]$. If there exists $1 \leq p \leq t$ such that

$$\{i_1, j_1, i_2, j_2, \dots, i_t, j_t\} \setminus \{i_p, j_p\} = \{1, 2, \dots, n+1\},$$

since $[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_{p-1},j_{p-1}}]$ is normal, we have

$$[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_{p-1},j_{p-1}}, R_{i_p,j_p}] \leq [[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_{p-1},j_{p-1}}].$$

(If $p = 1$, then we use $[R_{i_1,j_1}, R_{i_2,j_2}] \leq R_{i_2,j_2}$.) It follows that

$$[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}] \leq [[R_{i_1,j_1}, R_{i_2,j_2}], \dots, \hat{R}_{i_p,j_p}, \dots, R_{i_t,j_t}]$$

with $\{i_1, j_1, i_2, j_2, \dots, i_t, j_t\} \setminus \{i_p, j_p\} = \{1, 2, \dots, n+1\}$. By repeating this process by removing surplus entries, we have

$$[[R_{i_1,j_1}, R_{i_2,j_2}], \dots, R_{i_t,j_t}] \leq H$$

and hence the result. \square

The following simple result is well-known and follows from the structure of normal forms of free products with amalgamation (for the proof see, for example, [10]):

Lemma 3.7. *Let $G = G_1 *_A G_2$ be a free product with amalgamation such that $G_1 \neq A$ and $G_2 \neq A$. Then $Z(G) \leq Z(A)$.*

3.5. Proof of Theorem 1.2. We use our simplicial group model $\mathcal{T}(S^k, \alpha)$ for ΩS^k . Consider the construction of the subgroup $Q_{n,k}$ of P_n . By the definition of the simplicial group AP_* , the iterated degeneracy operations on $\text{AP}_1 = P_2$ are given by the cabling and so the elements x_1, \dots, x_{k-2} in Step 1 are the canonical basis for the subgroup

$$\Theta(F[S^1]_{k-2}) \leq \text{AP}_{k-2} = P_{k-1}.$$

Since $d_i x_i = d_i x_{i+1}$ for $1 \leq i \leq k-3$ and $d_0 x_1 = d_{k-2} x_{k-2} = 1$, we have $d_i \alpha_k = 1$ for $0 \leq i \leq k-2$. It follows that α_k is a Moore cycle in $F[S^1]_{k-2}$ with $\alpha_k \neq 1$. The elements y_j , $1 \leq j \leq \binom{n-1}{k-2}$, are standard basis for the subgroup

$$\phi_{\alpha_k}(F[S^{k-2}]_{n-1}) \leq \text{AP}_{n-1} = P_n$$

since they are obtained by cabling on α_k . It follows that

$$P_n *_{Q_{n,k}} P_n = (\text{AP}_* *_{F[S^{k-2}]} \text{AP}_*)_{n-1} = \mathcal{T}(S^k; \alpha_k)_{n-1}.$$

Theorem 1.2 is a special case of the following slightly more general statement.

Theorem 3.8. *Let $k \geq 3$ and let $\alpha \neq 1 \in F[S^1]_{k-2}$ be a Moore cycle. Then the simplicial group $\mathcal{T}(S^k; \alpha) \simeq \Omega S^k$ has the following properties:*

- 1) *In the group $\mathcal{T}(S^k; \alpha)_{n-1} = P_n *_{F[S^{k-2}]_{n-1}} P_n$, the Moore boundaries*

$$\mathcal{B}_{n-1} \mathcal{T}(S^k; \alpha) = [R_{i,j} \mid 1 \leq i < j \leq n]_S.$$

- 2) *The homotopy group $\pi_n(S^k) \cong \pi_{n-1}(\Omega S^k) \cong \pi_{n-1}(\mathcal{T}(S^k; \alpha))$ is isomorphic to the center of the group*

$$\mathcal{T}(S^k; \alpha)_{n-1} / \mathcal{B}_{n-1} \mathcal{T}(S^k; \alpha) = (P_n *_{F[S^{k-2}]_{n-1}} P_n) / [R_{i,j} \mid 1 \leq i < j \leq n]_S.$$

for any n if $k > 3$ and any $n \neq 3$ if $k = 3$.

Proof. (1). By definition, the simplicial group $\mathcal{T}(S^k; \alpha)$ is given by the free product with amalgamation $\text{AP}_* *_{F[S^{k-2}]} \text{AP}_*$. Thus $\mathcal{T}(S^k; \alpha)$ is a simplicial quotient group of the free product $\text{AP}_* * \text{AP}_*$. Let \mathcal{G} be the simplicial group given in Subsection 3.4. Then AP_* is a simplicial quotient group of \mathcal{G} . It follows that there is a simplicial epimorphism

$$g: \mathcal{G} * \mathcal{G} \longrightarrow \mathcal{T}(S^k; \alpha).$$

By Proposition 2.1,

$$N(g) = g|: N_n(\mathcal{G} * \mathcal{G}) \longrightarrow N_n(\mathcal{T}(S^k; \alpha))$$

is an epimorphism and so

$$\begin{aligned} \mathcal{B}_{n-1}(\mathcal{T}(S^k; \alpha)) &= d_0(N_n(\mathcal{T}(S^k; \alpha))) \\ &= d_0(g(N_n(\mathcal{G} * \mathcal{G}))) \\ &= g(d_0(N_n(\mathcal{G} * \mathcal{G}))) \\ &= g(\mathcal{B}_{n-1}(\mathcal{G} * \mathcal{G})). \end{aligned}$$

Assertion (1) follows from Lemma 3.5.

(2). **Case I.** $k > 3$. Since $\mathcal{T}(S^k; \alpha)_q = P_{q+1} *_{F[S^{k-2}]_q} P_{q+1}$ is a free product with amalgamation, the center $Z(\mathcal{T}(S^k; \alpha)_q) \leq Z(F[S^{k-2}]_q) = \{1\}$ for $q \geq k-1$ by Lemma 3.7. For $q = k-2$, then $Z(\mathcal{T}(S^k; \alpha)_{k-2}) \leq F[S^{k-2}]_{k-2} = \langle \alpha \rangle = \mathbb{Z}$ by Lemma 3.7. Since α is Moore cycle, α is a Brunnian braid in P_{k-1} . Recall that the center of P_{k-1} is given by the full-twist braid Δ^2 [4] with the property that, by removing any one of the strands of Δ^2 , it becomes a generator for the center of P_{k-1} and $d_i \Delta^2 \neq 1$ for $k > 3$. Since α is Brunnian braid, any power $\alpha^m \notin Z(P_{k-1})$ for $m \neq 0$. It follows that $\alpha^m \notin Z(P_{k-1} *_{F[S^{k-2}]_{k-2}} P_{k-1})$ for $m \neq 0$. Thus $Z(\mathcal{T}(S^k; \alpha)_{k-2}) = \{1\}$. For $q < k-2$, $\mathcal{T}(S^k, \alpha)_q$ is a free product and so $Z(\mathcal{T}(S^k, \alpha)_q) = \{1\}$, where for the low cases, $\mathcal{T}(S^k; \alpha)_1 = P_2 * P_2$ is a free group of rank 2 and $\mathcal{T}(S^k; \alpha)_0 = \{1\}$. Thus the center $Z(\mathcal{T}(S^k; \alpha)_q) = \{1\}$ for all $q \geq 0$. It follows from [22, Proposition 2.14] that

$$\pi_q(\mathcal{T}(S^k; \alpha)) \cong Z(\mathcal{T}(S^k; \alpha)_q) / \mathcal{B}_q(\mathcal{T}(S^k; \alpha))$$

for $q \geq 1$. This isomorphism also holds for $q = 0$ because $\mathcal{T}(S^k; \alpha)_0 = \{1\}$.

Case II. $k = 3$. By the same arguments as above, we have $Z(\mathcal{T}(S^3; \alpha)_q) = \{1\}$ for $q \geq 2$. By [22, Proposition 2.14], we have

$$(3.9) \quad \pi_q(\mathcal{T}(S^3; \alpha)) \cong Z(\mathcal{T}(S^3; \alpha)_q) / \mathcal{B}_q(\mathcal{T}(S^3; \alpha))$$

for $q \geq 3$. We only need to check that this isomorphism also holds for the cases $q = 0, 1$. (The case that $q = 2$ is the exceptional case, which is excluded in the statement.) When $q = 0$, both sides are trivial groups. Consider the case $q = 1$. Note that $AP_1 = P_2 \cong \mathbb{Z}$ generated by A_{12} . Since α is not trivial, it is given by a nontrivial power of A_{12} . Let $\alpha = A_{12}^m$ for some $m \neq 0$. Then $\mathcal{T}(S^3; \alpha)_1$ is given by the pushout diagram

$$\begin{array}{ccc} P_2 = \mathbb{Z} & \xrightarrow{m} & P_2 = \mathbb{Z} \\ \downarrow m & & \downarrow \\ P_2 = \mathbb{Z} & \longrightarrow & \mathcal{T}(S^3; \alpha)_1. \end{array}$$

Since $R_{1,2} = \langle A_{1,2}, A'_{1,2} \rangle^{\mathcal{T}(S^3; \alpha)_1} = \mathcal{T}(S^3; \alpha)_1$ because $\mathcal{T}(S^3; \alpha)_1$ is generated by $A_{1,2}$ and $A'_{1,2}$, we have

$$\mathcal{B}_1(\mathcal{T}(S^3; \alpha)) = \mathcal{T}(S^3; \alpha)_1$$

and so

$$Z(\mathcal{T}(S^3; \alpha)_1) / \mathcal{B}_1(\mathcal{T}(S^3; \alpha)) = \mathcal{T}(S^3; \alpha)_1 / \mathcal{B}_1(\mathcal{T}(S^3; \alpha)) = \{1\}.$$

On the other hand,

$$\pi_1(\mathcal{T}(S^3; \alpha)) = \pi_1(\Omega S^3) = \pi_2(S^3) = \{1\}.$$

Thus isomorphism (3.9) holds for $q = 1$. This finishes the proof. \square

Example 3.9. In this example, we discuss the exceptional case by determining the center of the group:

$$G = (P_3 *_{F[S^1]_2} P_3) / [R_{i,j} \mid 1 \leq i < j \leq 3]_S,$$

where $\alpha = A_{1,2}^m$ with some $m \neq 0$. By definition, the subgroup $F[S^1]_2 \leq P_3$ is generated by $x_1 = s_1 \alpha_3 = (A_{1,3} A_{2,3})^m$ and $x_2 = s_0 \alpha_3 = (A_{1,2} A_{1,3})^m$. Thus the

free product with amalgamation $P_3 *_F[S^1]_2 P_3$ is given as the quotient group of $P_3 * P_3$ by the new relations:

$$(3.10) \quad (A_{1,3}A_{2,3})^m = (A'_{1,3}A'_{2,3})^m \text{ and } (A_{1,2}A_{1,3})^m = (A'_{1,2}A'_{1,3})^m.$$

Consider the subgroup $[R_{i,j} \mid 1 \leq i < j \leq 3]_S$ of $P_3 *_F[S^1]_2 P_3$. Observe that

$$[A_{1,2}, A_{1,3}], [A_{1,2}, A_{2,3}], [A_{1,3}, A_{2,3}] \in [R_{i,j} \mid 1 \leq i < j \leq 3]_S,$$

the subgroup $\langle A_{1,2}, A_{1,3}, A_{2,3} \rangle$ is abelian in G . Similarly the subgroup $\langle A'_{1,2}, A'_{1,3}, A'_{2,3} \rangle$ is abelian in G . Thus $(A'_{1,2}A'_{1,3})^m = (A'_{1,2})^m(A'_{1,3})^m$ in G and from equation (3.10)

$$(A'_{1,2})^m = (A_{1,2})^m(A_{1,3})^m(A'_{1,3})^{-m}.$$

It follows that $A'_{1,2}$ commutes with $A_{1,2}$ since $A_{1,2}$ commutes with $(A_{1,2})^m, (A_{1,3})^m$ and $(A'_{1,3})^m$. From this, we conclude that $(A'_{1,2})^m \in Z(G)$ because $A'_{1,2}$ commutes with all of the generators for G . Similarly $(A_{1,2})^m, (A_{1,3})^m, (A_{2,3})^m, (A'_{1,3})^m, (A'_{2,3})^m \in Z(G)$. Thus the subgroup

$$(3.11) \quad H = \langle (A_{1,2})^m, (A_{1,3})^m, (A_{2,3})^m, (A'_{1,2})^m, (A'_{1,3})^m, (A'_{2,3})^m \rangle \leq Z(G).$$

Let

$$G' = (\mathbb{Z}(A_{1,2})/m * \mathbb{Z}(A'_{1,2})/m) \times (\mathbb{Z}(A_{1,3})/m * \mathbb{Z}(A'_{1,3})/m) \times (\mathbb{Z}(A_{2,3})/m * \mathbb{Z}(A'_{2,3})/m)$$

and let $\phi: P_3 * P_3 \rightarrow G'$ be the canonical quotient homomorphism defined by sending generators to generators. Then

$$\phi(x_1) = \phi(x_2) = 1.$$

Moreover $\phi([R_{i,j} \mid 1 \leq i < j \leq 3]_S) = 1$ with $\phi(H) = 1$ and so ϕ induces an epimorphism $\bar{\phi}$ in the following diagram:

$$\begin{array}{ccc} P_3 * P_3 & \xrightarrow{\phi} & G' = (\mathbb{Z}/m * \mathbb{Z}/m) \times (\mathbb{Z}/m * \mathbb{Z}/m) \times (\mathbb{Z}/m * \mathbb{Z}/m) \\ \downarrow q & \nearrow \bar{\phi} & \\ G/H & & \end{array}$$

On the other hand, the group homomorphism

$$\mathbb{Z}(A_{1,2}) * \mathbb{Z}(A'_{1,2}) \longrightarrow G/H$$

factors through the quotient $\mathbb{Z}(A_{1,2})/m * \mathbb{Z}(A'_{1,2})/m$. Similarly there are canonical group homomorphisms from $\mathbb{Z}(A_{1,3})/m * \mathbb{Z}(A'_{1,3})/m$ and $\mathbb{Z}(A_{2,3})/m * \mathbb{Z}(A'_{2,3})/m$ to G/H . Since the subgroup $\langle A_{1,2}, A'_{1,2} \rangle$, $\langle A_{1,3}, A'_{1,3} \rangle$ and $\langle A_{2,3}, A'_{2,3} \rangle$ commute with each other in the group G , there is a group epimorphism

$$\psi: G' \rightarrow G/H$$

such that $\bar{\phi} \circ \psi = \text{id}_{G'}$ since all of generators of G/H lie in the image of ψ . It follows that

$$G/H \cong G' = (\mathbb{Z}/m * \mathbb{Z}/m) \times (\mathbb{Z}/m * \mathbb{Z}/m) \times (\mathbb{Z}/m * \mathbb{Z}/m).$$

Since $Z(G') = \{1\}$, $Z(G/H) = \{1\}$ and so

$$Z(G) \leq H.$$

Together with equation (3.11), we have $Z(G) = H \cong \mathbb{Z}^{\oplus 4}$. □

4. DESCRIPTION OF HOMOTOPY GROUPS OF THE MOORE SPACES $M(\mathbb{Z}/q, k)$ WITH $k \geq 3$

In this section, we give an explicit combinatorial description of the homotopy groups of the Moore spaces $M(\mathbb{Z}/q, k)$ with $k \geq 3$. This description highlights our methodology for giving combinatorial descriptions of homotopy groups using free products of braid groups.

4.1. An Embedding of $F[S^{k-1}]$ into $\mathcal{T}(S^k; \alpha)$ for Moore Boundaries α . Let $\tilde{\alpha} \in N_{k-1}F[S^1]$ with $d_0\tilde{\alpha} \neq 1$. We are going to construct a simplicial monomorphism $F[S^{k-1}] \rightarrow \mathcal{T}(S^k; d_0\tilde{\alpha})$, which is also a homotopy equivalence.

Let

$$f_{\tilde{\alpha}}: \Delta[k-1] \longrightarrow F[S^1]$$

be the representing map of the element $\tilde{\alpha}$ with $f_{\tilde{\alpha}}(\sigma_k) = \tilde{\alpha}$, where $\sigma_k = (0, 1, \dots, k-1) \in \Delta[k-1]$. Let $\Lambda^0[k-1]$ be the simplicial subset of $\Delta[k-1]$ generated by $d_j\sigma_{k-1}$ for $j > 0$ and let

$$\bar{\Delta}[k-1] = \Delta[k-1]/\Lambda^0[k-1].$$

Since $d_j\tilde{\alpha} = 1$ for $j > 0$, the simplicial map $f_{\tilde{\alpha}}$ factors through the simplicial quotient $\bar{\Delta}[k-1]$. Let

$$(4.1) \quad \bar{f}_{\tilde{\alpha}}: \bar{\Delta}[k-1] \longrightarrow F[S^1]$$

be the resulting simplicial map with $\bar{f}_{\tilde{\alpha}}(\sigma_{k-1}) = \tilde{\alpha}$. By the universal property of Milnor's construction, there exists a unique simplicial homomorphism

$$(4.2) \quad \theta_{\tilde{\alpha}}: F[\bar{\Delta}[k-1]] \longrightarrow F[S^1]$$

such that $\theta_{\tilde{\alpha}}|_{\bar{\Delta}[k-1]} = \bar{f}_{\tilde{\alpha}}$.

Lemma 4.1. *The simplicial group $F[\bar{\Delta}[k-1]]$ is contractible and the map*

$$\theta_{\tilde{\alpha}}: F[\bar{\Delta}[k-1]] \longrightarrow F[S^1]$$

is a simplicial monomorphism.

Proof. Recall [7] that the geometric realization $|\Delta[k-1]|$ is the standard $(k-1)$ -simplex Δ^{k-1} and $|\Lambda^0[k]|$ is the union of all faces of Δ^{k-1} except the first face. Thus both $|\Delta[k-1]|$ and $|\Lambda^0[k]|$ are contractible and so is $|\bar{\Delta}[k-1]| = |\Delta[k-1]|/\Lambda^0[k-1]$. It follows that

$$|F[\bar{\Delta}[k-1]]| \simeq \Omega\Sigma|\bar{\Delta}[k-1]|$$

is contractible.

The proof of the statement regarding $\theta_{\tilde{\alpha}}$ is similar to that of Lemma 3.2. The image $\theta_{\tilde{\alpha}}(F[\bar{\Delta}[k-1]])$ is a simplicial free group because it is a simplicial subgroup of the simplicial free group $F[S^1]$. Following the lines in the proof of Lemma 3.2, for checking that $\theta_{\tilde{\alpha}}: F[\bar{\Delta}[k-1]] \rightarrow \theta_{\tilde{\alpha}}(F[\bar{\Delta}[k-1]])$ is a simplicial monomorphism, it suffices to show that

$$N\theta_{\tilde{\alpha}}^{\text{ab}}: N_*F[\bar{\Delta}[k-1]]^{\text{ab}} \longrightarrow N_*\theta_{\tilde{\alpha}}(F[\bar{\Delta}[k-1]])$$

is an isomorphism. This follows directly from the computations that

$$N_qF[\bar{\Delta}[k-1]]^{\text{ab}} = \begin{cases} \mathbb{Z}(\sigma_{k-1}) & \text{for } q = k-1, \\ \mathbb{Z}(d_0\sigma_{k-1}) & \text{for } q = k-2, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_q \theta_{\tilde{\alpha}}(F[\bar{\Delta}[k-1]])^{\text{ab}} = \begin{cases} \mathbb{Z}(\tilde{\alpha}) & \text{for } q = k-1, \\ \mathbb{Z}(d_0 \tilde{\alpha} = \alpha) & \text{for } q = k-2, \\ 0 & \text{otherwise} \end{cases}$$

and $\theta_{\tilde{\alpha}}(\sigma_{k-1}) = \tilde{\alpha}$. □

Now from the above Lemma, the simplicial monomorphism

$$\phi_{\alpha}: F[S^{k-2}] \longrightarrow \text{AP}_*$$

is given by the composite

$$F[S^{k-2}] \xrightarrow{\iota} F[\bar{\Delta}[k-1]] \xrightarrow{\theta_{\tilde{\alpha}}} F[S^1] \xrightarrow{\Theta} \text{AP}_*.$$

It follows that $\Theta \circ \theta_{\tilde{\alpha}}: F[\bar{\Delta}[k-1]] \rightarrow \text{AP}_*$ induces a simplicial monomorphism

$$(4.3) \quad F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]] \hookrightarrow \text{AP}_* *_{F[S^{k-2}]} \text{AP}_*,$$

which is a homotopy equivalence by Theorem 2.2. Let σ'_{k-1} denote the element σ_{k-1} in second copy of $F[\bar{\Delta}[k-1]]$ in the free product with amalgamation $F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]$. Let

$$z_{k-1} = \sigma_{k-1}(\sigma'_{k-1})^{-1} \in (F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]])_{k-1}.$$

Then z_{k-1} is a Moore cycle because

$$d_j z_{k-1} = d_j \sigma_{k-1} (d_j \sigma'_{k-1})^{-1} = 1$$

for $j > 0$ in $F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]$ and

$$d_0 z_{k-1} = d_0 \sigma_{k-1} (d_0 \sigma'_{k-1})^{-1} = 1$$

since $d_0 \sigma_{k-1} = d_0 \sigma'_{k-1}$ lies in the amalgamated subgroup $F[S^{k-2}]$. Let $f_{z_{k-1}}: S^{k-1} \rightarrow F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]$ be the representing map of z_{k-1} and let

$$\tilde{f}_{z_{k-1}}: F[S^{k-1}] \longrightarrow F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]$$

be the simplicial homomorphism induced by $f_{z_{k-1}}$.

Lemma 4.2. *Let $\tilde{f}_{z_{k-1}}$ be defined as above. Then*

- 1) $\tilde{f}_{z_{k-1}}$ is a simplicial monomorphism.
- 2) $\tilde{f}_{z_{k-1}}$ is a homotopy equivalence.

Proof. (1). Observe that

$$F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]] = F[\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]]$$

is a simplicial free group, where $\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]$ is the simplicial union by identification $d_0 \sigma_{k-1}$ with $d_0 \sigma'_{k-1}$. Assertion (1) follows from the lines of the proof of Lemma 3.2.

(2). Since

$$\tilde{f}_{z_{k-1}}: F[S^{k-1}] \simeq \Omega S^k \longrightarrow F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]] \simeq \Omega S^k$$

is a simplicial homomorphism, it is a loop map. Thus it suffices to show that $\tilde{f}_{z_{k-1}}$ induces an isomorphism

$$\tilde{f}_{z_{k-1}*}: \pi_{k-1}(F[S^{k-1}]) \cong \mathbb{Z} \longrightarrow \pi_{k-1}(F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]) \cong \mathbb{Z}.$$

Note that

$$\pi_{k-1}(F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]) \cong \pi_{k-1}(F[\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]]^{\text{ab}}).$$

Now the Moore chain complex of $F[\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]]^{\text{ab}}$ is given by

$$N_q F[\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]]^{\text{ab}} = \begin{cases} \mathbb{Z}(\sigma_{k-1}) \oplus \mathbb{Z}(\sigma'_{k-1}) & \text{for } q = k-1, \\ \mathbb{Z}(d_0 \sigma_{k-1} = d_0 \sigma'_{k-1}) & \text{for } q = k-2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\pi_{k-1}(F[\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]]^{\text{ab}})$ is generated by $\sigma_{k-1} - \sigma'_{k-1}$, which is the image of z_{k-1} in the abelianization $F[\bar{\Delta}[k-1] \cup \bar{\Delta}[k-1]]^{\text{ab}}$. It follows that

$$\tilde{f}_{z_{k-1}*} : \pi_{k-1}(F[S^{k-1}]) \longrightarrow \pi_{k-1}(F[\bar{\Delta}[k-1]] *_{F[S^{k-2}]} F[\bar{\Delta}[k-1]]).$$

is an isomorphism and hence the result. \square

4.2. Description for $\pi_*(M(\mathbb{Z}/q, k))$ with $k \geq 3$. With the preparation in the previous subsection, we can now construct a simplicial group model for $\Omega M(\mathbb{Z}/q, k)$ with $k \geq 3$. Let $\alpha \in \mathcal{Z}_{k-1} F[S^1]$ be a Moore cycle with $\alpha \neq 1$ and let $\tilde{\alpha} \in N_{k-1} F[S^1]$ be a Moore chain such that $d_0 \tilde{\alpha} \neq 1$. From Lemma 4.2 together with isomorphism (4.3), there is a simplicial monomorphism

$$\delta_{\tilde{\alpha}} : F[S^{k-1}] \longrightarrow \mathcal{T}(S^k; d_0 \tilde{\alpha}),$$

which is a homotopy equivalence. Let

$$F[q] : F[S^{k-1}] \longrightarrow F[S^{k-1}]$$

be the simplicial homomorphism such that

$$F[q](x) = x^q$$

for $x \in S^{k-1} \subseteq F[S^{k-1}]$. Clearly $F[q]$ is a simplicial monomorphism. Now define the simplicial group $\mathcal{T}(M(\mathbb{Z}/q, k); \alpha)$ to be the free product with amalgamation

$$\begin{array}{ccc} F[S^{k-1}] & \xrightarrow{\delta_{\tilde{\alpha}} \circ F[q]} & \mathcal{T}(S^k; d_0 \tilde{\alpha}) \\ \downarrow \phi_{\alpha} & & \downarrow \\ \text{AP}_* & \longrightarrow & \mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha) = \mathcal{T}(S^k; d_0 \tilde{\alpha}) *_{F[S^{k-1}]} \text{AP}_*. \end{array}$$

The construction of $\delta_{\tilde{\alpha}} \circ F[q]$ is explicitly given as follows:

Regard $\tilde{\alpha}$ as in k -strand braid through the embedding $\Theta : F[S^1] \rightarrow \text{AP}_$. Let $\tilde{\alpha}'$ be a copy of $\tilde{\alpha}$ for the second copy of AP_* in the free product with amalgamation*

$$\mathcal{T}(S^k; d_0 \tilde{\alpha}) = \text{AP}_* *_{F[S^{k-2}]} \text{AP}_*.$$

Let σ_{k-1} be the non-degenerate element in S_{k-1}^{k-1} . Then

$$\delta_{\tilde{\alpha}} \circ F[q] : F[S^{k-1}] \rightarrow \mathcal{T}(S^k; d_0 \tilde{\alpha})$$

is the unique simplicial homomorphism such that $\delta(\sigma_{k-1}) = (\tilde{\alpha}(\tilde{\alpha}')^{-1})^q$. In the language of braids, $\delta_{\tilde{\alpha}} \circ F[q](F[S^{k-1}])$ is the subgroup of $\mathcal{T}(S^k; d_0 \tilde{\alpha}) = \text{AP}_ *_{F[S^{k-2}]} \text{AP}_*$ generated by the cablings of $(\tilde{\alpha}(\tilde{\alpha}')^{-1})^q$ in the self free product with amalgamation of braid groups.*

One interesting point in the simplicial group

$$\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha) = (\mathrm{AP}_* *_{F[S^{k-2}]} \mathrm{AP}_*) *_{F[S^{k-1}]} \mathrm{AP}_*$$

is that we identify the q -th power $(\tilde{\alpha}(\tilde{\alpha}')^{-1})^q \in \mathrm{AP}_* *_{F[S^{k-2}]} \mathrm{AP}_*$ with $\alpha \in \mathrm{AP}_*$. So the cablings of α have q -th roots in $\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)$.

Theorem 4.3. *Let $\alpha \in \mathcal{Z}_{k-1}F[S^1]$ be a Moore cycle with $\alpha \neq 1$ and let $\tilde{\alpha} \in N_{k-1}F[S^1]$ be a Moore chain such that $d_0\tilde{\alpha} \neq 1$. Then the simplicial group $\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)$ is homotopy equivalent to the loop space $\Omega M(\mathbb{Z}/q, k)$ of the Moore space. Moreover the canonical inclusion*

$$\mathcal{T}(S^k; d_0\tilde{\alpha}) \hookrightarrow \mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)$$

is homotopic to the looping of the inclusion $S^k \hookrightarrow M(\mathbb{Z}/q, k)$.

Proof. By Theorem 2.2, the classifying space $\bar{W}(\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha))$ is given by the homotopy push-out

$$\begin{array}{ccc} S^k & \xrightarrow{\bar{W}(\delta_{\tilde{\alpha}} \circ F[q])} & S^k \simeq \bar{W}(S^k; d_0\tilde{\alpha}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bar{W}(\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)). \end{array}$$

Since

$$\bar{W}(\delta_{\tilde{\alpha}} \circ F[q]_*): \pi_k(S^k) \cong \pi_{k-1}(F[S^{k-1}]) \longrightarrow \pi_k(S^k) \cong \pi_{k-1}(\mathcal{T}(S^k; d_0\tilde{\alpha}))$$

is of degree q , $\bar{W}(\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)) \simeq M(\mathbb{Z}/q, k)$. Observe that the right column of above diagram is homotopic to the inclusion of the bottom cell $S^k \hookrightarrow M(\mathbb{Z}/q, k)$. The assertions follow. \square

Let $A_{i,j}$, $A'_{i,j}$ and $A''_{i,j}$ be copies of $A_{i,j}$ for generators for P_n in the free product with amalgamation

$$\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)_{n-1} = (P_n *_{F[S^{k-2}]}_{n-1} P_n) *_{F[S^{k-1}]}_{n-1} P_n$$

and let $R_{i,j}$ be the normal closure of $A_{i,j}$, $A'_{i,j}$ and $A''_{i,j}$ in $\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)_{n-1}$.

Theorem 4.4. *Let $k \geq 3$. Let $\alpha \in \mathcal{Z}_{k-1}F[S^1]$ be a Moore cycle with $\alpha \neq 1$ and let $\tilde{\alpha} \in N_{k-1}F[S^1]$ be a Moore chain such that $d_0\tilde{\alpha} \neq 1$. Then $\pi_n(M(\mathbb{Z}/q, k))$ is isomorphic to the center of the group*

$$((P_n *_{F[S^{k-2}]}_{n-1} P_n) *_{F[S^{k-1}]}_{n-1} P_n) / [R_{i,j} \mid 1 \leq i < j \leq n]_S$$

for any n .

Proof. Since $\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha)$ is a simplicial quotient group of $\mathcal{G} * \mathcal{G} * \mathcal{G}$, the Moore boundaries

$$\mathcal{B}_{n-1}\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}, \alpha) = [R_{i,j} \mid 1 \leq i < j \leq n]_S$$

by Lemma 3.5. Observe that the group $(P_m *_{F[S^{k-2}]}_{m-1} P_m) *_{F[S^{k-1}]}_{m-1} P_m$ has trivial center by Lemma 3.7. The assertion follows from [22, Proposition 2.14]. \square

Remark 4.5. An explicit choice of α and $\tilde{\alpha}$ can be given. For instance, we can choose

$$\alpha_{k+1} = [[[x_1^{-1}, x_1 x_2^{-1}], x_2 x_3^{-1}], \dots, x_{k-2} x_{k-1}^{-1}, x_{k-1}]$$

in Theorem 1.2 as a k -strand Brunnian braid and choose

$$\tilde{\alpha}_k = [[x_1 x_2^{-1}, x_2 x_3^{-1}], \dots, x_{k-2} x_{k-1}^{-1}, x_{k-1}]$$

as a k -strand quasi-Brunnian braid in the sense of [5]. Then we obtain an explicit simplicial group model $\mathcal{T}(M(\mathbb{Z}/q, k); \tilde{\alpha}_k, \alpha_{k+1})$ for $\Omega M(\mathbb{Z}/q, k)$. \square

5. DESCRIPTION OF THE HOMOTOPY GROUPS OF MOORE SPACES $M(\mathbb{Z}/q, 2)$ AND PROOF OF THEOREM 1.3

Let $\mathcal{T}(M(\mathbb{Z}/q, 2))$ be the free product with amalgamation by the following diagram

$$\begin{array}{ccc} F[S^1] & \xrightarrow{F[q]} & F[S^1] \\ \downarrow \Theta & & \downarrow \\ \mathrm{AP}_* & \longrightarrow & \mathcal{T}(M(\mathbb{Z}/q, 2)) = \mathrm{AP}_* *_{F[S^1]} F[S^1]. \end{array}$$

By Theorem 2.2, there is a homotopy push-out

$$\begin{array}{ccc} S^2 \simeq \bar{W}F[S^1] & \xrightarrow{\bar{F}[q] \simeq [q]} & S^2 \simeq \bar{W}F[S^1] \\ \downarrow \Theta & & \downarrow \\ \bar{W}\mathrm{AP}_* \simeq * & \longrightarrow & \bar{W}\mathcal{T}(M(\mathbb{Z}/q, 2)) \end{array}$$

and so $\bar{W}\mathcal{T}(M(\mathbb{Z}/q, 2)) \simeq M(\mathbb{Z}/q, 2)$. Namely $\mathcal{T}(M(\mathbb{Z}/q, 2))$ is a simplicial group model for $\Omega M(\mathbb{Z}/q, 2)$.

For each n , the homomorphism

$$F[q]: F[S^1]_{n-1} = F_{n-1} \longrightarrow F[S^1]_{n-1} = F_{n-1}$$

is the homomorphism ϕ_q described in Theorem 1.3. Thus as a group

$$\mathcal{T}(M(\mathbb{Z}/q, 2))_{n-1} = P_n *_{\phi_q} F_{n-1}.$$

We give an more explicit description of the group $\mathcal{T}(M(\mathbb{Z}/q, 2))_{n-1}$ using degeneracy operations. Let $\{x_j\}_{j=1, \dots, n-1}$ be the set of generators for $F_{n-1} = F[S^1]_{n-1}$ as the second factor in the free product $P_n *_{\phi_q} F_{n-1}$ for $1 \leq j \leq n-1$. (**Note.** In the introduction to Theorem 1.3, we write y_j for x_j .) As an element in $F[S^1]_{n-1}$,

$$x_j = s_{n-2} \cdots s_{j+1} s_j s_{j-2} s_{j-3} \cdots s_1 s_0 \sigma_1$$

for $1 \leq j \leq n-1$. The group $\mathcal{T}(M(\mathbb{Z}/q, 2))_{n-1}$ is the quotient group of $P_n * F_{n-1}$ by the relation

$$s_{j+1} s_j s_{j-2} s_{j-3} \cdots s_1 s_0 A_{1,2} = x_j^q$$

for $1 \leq j \leq n-1$, where $s_{j+1}s_js_{j-2}s_{j-3}\cdots s_1s_0A_{1,2}$ the cabling of $A_{1,2}$ as the picture in the introduction.

Let $z_1 = x_1$, $z_n = x_{n-1}$ and $z_i = x_i x_{i-1}^{-1}$, for $i = 2, \dots, n-1$. Now let $R_i = \langle z_i \rangle^{P_n *_{\phi_q} F_{n-1}}$ be the normal closure of z_i in $P_n *_{\phi_q} F_{n-1}$ for $1 \leq i \leq n$ and let $R_{s,t} = \langle A_{s,t} \rangle^{P_n *_{\phi_q} F_{n-1}}$ be the normal closure of $A_{s,t}$ in $P_n *_{\phi_q} F_{n-1}$ for $1 \leq s < t \leq n$. Define the index set $\text{Index}(R_j) = \{j\}$ for $1 \leq j \leq n$ and $\text{Index}(R_{s,t}) = \{s, t\}$ for $1 \leq s < t \leq n$. Now define the symmetric commutator subgroup

$$[R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S = \prod_{\{1,2,\dots,n\} = \bigcup_{j=1}^t \text{Index}(C_j)} [[C_1, C_2], \dots, C_t],$$

where each $C_j = R_i$ or $R_{s,t}$ for some i or (s, t) .

Theorem 5.1 (Theorem 1.3). *The homotopy group $\pi_n(M(\mathbb{Z}/q, 2))$ is isomorphic to the center of the group*

$$(P_n *_{\phi_q} F_{n-1}) / [R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S$$

for any $n > 3$.

Proof. The proof is similar to that of Theorem 1.2. It is easy to see that the group $\mathcal{T}(M(\mathbb{Z}/q, 2))_m = P_{m+1} *_{\phi_q} F_m$ has the trivial center for $m \geq 2$. From [22, Proposition 2.14], $\pi_m(\mathcal{T}(M(\mathbb{Z}/q, 2))) \cong \pi_{m+1}(M(\mathbb{Z}/q, 2))$ is isomorphic to the center of $\mathcal{T}(M(\mathbb{Z}/2))_m / \mathcal{B}_m \mathcal{T}(M(\mathbb{Z}/q, 2))$ for $m \geq 3$. Thus the key point is to show the Moore boundaries

$$\mathcal{B}_{n-1} \mathcal{T}(M(\mathbb{Z}/q, 2)) = [R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S.$$

We construct a simplicial group \tilde{F} by \tilde{F}_{n-1} generated by the letters z_1, \dots, z_n with face operation

$$d_j z_k = \begin{cases} z_k & \text{for } k < j+1 \\ 1 & \text{for } k = j+1 \\ z_{k-1} & \text{for } k > j+1 \end{cases}$$

and degeneracy operations

$$s_j z_k = \begin{cases} z_k & \text{for } k < j+1 \\ z_{j+1} z_{j+2} & \text{for } k = j+1 \\ z_{k+1} & \text{for } k > j+1 \end{cases}$$

for $0 \leq j \leq n-1$. Then \tilde{F} is a simplicial group with a simplicial epimorphism $f: \tilde{F} \rightarrow F[S^1]$ by sending the letter z_j of \tilde{F}_{n-1} to the element $z_j \in F[S^1]_{n-1}$. Let $g: \mathcal{G} \rightarrow \text{AP}_*$ be the canonical simplicial epimorphism. Then we have the simplicial epimorphism

$$\mathcal{G} * \tilde{F} \longrightarrow \text{AP}_* * F[S^1] \longrightarrow \mathcal{T}(M(\mathbb{Z}/q, 2)).$$

Observe that $\text{Ker}(d_n: (\mathcal{G} * \tilde{F})_n \rightarrow (\mathcal{G} * \tilde{F})_{n-1})$ is the normal closure of the elements $x_{i,n+1}, z_{n+1}$. By repeating the arguments in the proof of Lemma 3.5, we have

$$\mathcal{B}_{n-1}(\mathcal{G} * \tilde{F}) = [R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S$$

and hence the result. \square

Example. Consider the case $n = 3$. The group

$$G = (P_3 *_{\phi_q} F_2) / [R_i, R_{s,t} \mid 1 \leq i \leq 3, 1 \leq s < t \leq 3]_S$$

is given by generators $x_1, x_2, a_{12}, a_{13}, a_{23}$ and the following relations

$$\begin{aligned} x_1^g &= a_{12}a_{13}, \quad x_2^g = a_{13}a_{23}, \\ [[x_1^{g_1}, x_2^{g_2}], x_1] &= [[x_1^{g_1}, x_2^{g_2}], x_2] = 1, \quad g_1, g_2 \in G \\ [a_{12}^g, a_{13}] &= [a_{12}^g, a_{23}] = [a_{13}^g, a_{23}] = 1, \quad g \in G \\ [x_1^g, a_{23}] &= [(x_1x_2^{-1})^g, a_{13}] = [x_2^g, a_{12}] = 1, \quad g \in G. \end{aligned}$$

Presenting a_{13}, a_{23} via generators x_1, x_2, a_{12} , we get the following 3-generator presentation of G :

$$\begin{aligned} [[x_1^{g_1}, x_2^{g_2}], x_1] &= [[x_1^{g_1}, x_2^{g_2}], x_2] = 1, \quad g_1, g_2 \in G \\ [a_{12}^g, a_{12}^{-1}x_1^g] &= [a_{12}^g, x_1^{-g}a_{12}x_2^g] = [(a_{12}^{-1}x_1^g)^g, x_1^{-g}a_{12}x_2^g] = 1, \quad g \in G \\ [x_1^g, x_1^{-g}a_{12}x_2^g] &= [(x_1x_2^{-1})^g, a_{12}^{-1}x_1^g] = [x_2^g, a_{12}] = 1, \quad g \in G. \end{aligned}$$

Straightforward computations show that G is a 3-generator nilpotent group of class 2, given by generators x_1, x_2, a_{12} and relations

$$\begin{aligned} [a_{12}, x_2] &= [a_{12}, x_1^g] = [x_1^g, x_2^g] = [x_1, a_{12}x_2^g] = [x_1x_2^{-1}, a_{12}^{-1}x_1^g] = 1 \\ [[G, G], G] &= 1 \end{aligned}$$

It follows that the order of the element $[x_1, x_2]$ is $(2q, q^2)$ in G . The center of G is bigger than the subgroup generated by $[x_1, x_2]$, since a_{12}^q lies in the center. Denote $Z_1 = \langle a_{12}, x_1 \rangle^G$, $Z_2 = \langle a_{12}, x_1x_2^{-1} \rangle^G$, $Z_3 = \langle x_2 \rangle^G$. The homotopy group $\pi_3 M(Z/q, 2)$ is given now as the intersection

$$Z_1 \cap Z_2 \cap Z_3 \simeq \mathbb{Z}/(2q, q^2).$$

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